

Some observations on the Baireness of $C_k(X)$ for a locally compact space X

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Abstract

We prove some consistency results concerning the Moving Off Property for locally compact spaces, and thus the question of whether their function spaces are Baire.

1 Introduction

The Moving Off Property was introduced in [11] to characterize when $C_k(X)$ satisfies the Baire Category Theorem, for q -spaces X . Here we shall only be concerned with locally compact spaces (which are q), and so won't define q . We shall assume all spaces are Hausdorff.

Definition. *A moving off collection for a space X is a collection \mathcal{K} of non-empty compact sets such that for each compact L , there is a $K \in \mathcal{K}$ disjoint from L . A space satisfies the Moving Off Property (MOP) if each moving off collection includes an infinite subcollection with a discrete open expansion.*

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Theorem 1 [11]. *A locally compact space X satisfies the MOP if and only if $C_k(X)$ is Baire, i.e., satisfies the Baire Category Theorem.*

There is a less onerous equivalent of the MOP for locally compact spaces:

Lemma 2 [10]. *Let X be a locally compact space. Then X has the MOP if and only if every moving off collection for X includes an infinite discrete subcollection.*

We give a proof for the benefit of readers who are not topologists.

Proof. Let \mathcal{K} be a moving off collection for X . By local compactness, each $K \in \mathcal{K}$ can be fattened to an open set with compact closure. Let \mathcal{K}' be the collection of all compact closures of open sets around members of \mathcal{K} . Then \mathcal{K}' is moving off. For let C be a compact subset of X . There is a $K \in \mathcal{K}$ disjoint from C . By regularity and local compactness, there is an open $U \supseteq K$ with compact closure \overline{U} disjoint from C . Then $\overline{U} \in \mathcal{K}'$. Since we have established that \mathcal{K}' is moving off, by hypothesis it includes an infinite discrete collection $\{\overline{U}_n\}_{n < \omega}$. But each U_n included some $K_n \in \mathcal{K}$. Then $\{K_n\}_{n < \omega}$ is discrete and has the discrete open expansion $\{U_n\}_{n < \omega}$. \square

In [14], [15], and [24], assuming the existence of a supercompact cardinal, a model of set theory is constructed, which we shall refer to as a *model of* $PFA(S)[S]$. We refer the reader to those papers for a discussion of what $PFA(S)[S]$ is. In these papers various propositions concerning locally compact normal spaces are established in this model. We shall use:

Lemma 3 [15]. *In this model, locally compact hereditarily normal spaces which do not include a perfect pre-image of ω_1 are paracompact.*

Corollary 4 [14]. *In this model, locally compact, perfectly normal spaces are paracompact.*

Lemma 5 [24]. *In this model, locally compact normal spaces with Lindelöf number $\leq \aleph_1$ which do not include a perfect pre-image of ω_1 are paracompact.*

Let us also quote several useful results concerning the MOP.

Lemma 6 [17, 11]. *Countably compact spaces satisfying the MOP are compact.*

Lemma 7 [17, 11]. *First countable spaces satisfying the MOP are locally compact.*

Lemma 8 [17, 11]. *Locally compact, paracompact spaces satisfy the MOP.*

A stronger result is in Lemma 24 below

2 Locally compact, perfectly normal spaces and the MOP

Marion Scheepers asked us whether locally compact, perfectly normal spaces satisfy the MOP, and whether - if they do - they are paracompact. Here are the answers, modulo a supercompact cardinal.

Theorem 9. *There is a model of $PFA(S)[S]$ in which locally compact, perfectly normal spaces are paracompact and hence satisfy the MOP.*

Theorem 10. *There is a model in which there is a locally compact, perfectly normal space which does not satisfy the MOP.*

Proofs. Theorem 9 follows from Corollary 4 plus Lemma 8. Theorem 10 follows from Lemma 6, since *Ostaszewski's space* [18], constructed from \diamond , is locally compact, perfectly normal, countably compact, but not compact. \square

For the other question, obviously Corollary 4 answers it one way; for the other, we quote:

Lemma 11 [16]. *MA_{ω_1} implies there is a locally compact perfectly normal space with the MOP which is not paracompact.*

3 Counterexamples

Although the question of whether locally compact normal spaces with the MOP are paracompact has not been answered in ZFC, there are a number of consistent counterexamples which repurpose spaces familiar to normal Moore space fans. a)-f) are not collectionwise Hausdorff, hence not paracompact. Each is normal in some model.

- a) [16] The Cantor tree on a set of reals of size \aleph_1 is normal and has the MOP under MA_{ω_1} .

Definition. *A ladder system $\{\lambda_\alpha\}_{\alpha \in S}$, where S is a subset of some ordinal λ , is a set of sequences, where each λ_α is strictly increasing, converges to α , and has range disjoint from S . The corresponding ladder system space on $S \cup \bigcup \{\text{range } \lambda_\alpha : \alpha \in S\}$ has the points in each range λ_α isolated, while a basic open set about $\alpha \in S$ is $\{\alpha\} \cup$ a tail of λ_α .*

- b) [11] A ladder system space on a stationary subset of ω_1 has the MOP, and also is normal under MA_{ω_1} .

Note the first example is separable, while countable sets have countable closures in the second one.

- c) There is also a version of b) consistent with CH, indeed with \diamond . See [21], [4], [14].

We shall show that the idea of the proof of the MOP for b) (and hence c)) can be used to establish the MOP for:

- d) The tree topology on a special Aronszajn tree. This is known to be non-collectionwise Hausdorff, and to be normal under MA_{ω_1} [6].

as well as for the space of:

- e) Devlin and Shelah [2] isolate some points of a special Aronszajn tree and manage to force normality while keeping CH.

Generalizing the proof in [11] that a ladder system space on a stationary subset of ω_1 has the MOP, we obtain:

Theorem 12. *Suppose X is locally compact, locally countable, countable sets have countable closures, and $X = \bigcup_{\gamma < \omega_1} X_\gamma$, where each X_γ is countable, $X_\gamma \subsetneq X_{\gamma+1}$, and for γ a limit, $X_\gamma = \bigcup_{\alpha < \gamma} X_\alpha$. Further suppose that for γ a limit, for each x in the boundary of X_γ , there is a compact neighborhood $N(x)$ such that for each $\alpha < \gamma$, $N(x) \cap X_\alpha$ is compact. Then X has the MOP.*

Proof. Since countable sets have countable closures, without loss of generality we may assume that $\overline{X_\alpha} \subseteq X_{\alpha+1}$. Since compact sets are countable,

$$C = \{\alpha : x \in X_\alpha \text{ implies } N(x) \subseteq X_\alpha\}$$

is closed unbounded. Since X is first countable, each X_α has a countable base \mathcal{B}_α of compact sets open in X_α . For $\alpha \in C$, X_α is open, so these sets are open in X .

Let \mathcal{A} be a moving off collection for \mathcal{X} . For any $\alpha < \omega_1$, there is a countable ordinal $\delta(\alpha) \geq \alpha$ such that for $B \in \mathcal{B}_\alpha$, there is an $A \in \mathcal{A}$ such that $A \subseteq X_{\delta(\alpha)}$ and A is disjoint from B . Then

$$C' = \{\alpha \in C : \beta < \alpha \text{ implies } \delta(\beta) < \alpha\}$$

is closed unbounded. Take a strictly increasing sequence $\{\gamma_n\}_{n<\omega}$ in C' and let $\gamma = \sup_n \gamma_n$. Let $\{B_{m,k} : m < \omega\}$ enumerate the basic compact open sets of X_{γ_k} . Note $\bigcup\{B_{m,k} : m, k < \omega\}$ is a basis for X_γ . Let $\{x_{\gamma,i} : i < \omega\}$ enumerate $\overline{X_\gamma} - X_\gamma$. For each $j < \omega$, there is an $A_j \in \mathcal{A}$ with $A_j \subseteq X_{\gamma_j}$ and $A_j \cap \left(\bigcup_{k<j} A_k \cup \bigcup_{m,k<j} B_{m,k} \cup \bigcup_{n,i<j} (N(x_{\gamma,i}) \cap X_{\gamma_n})\right) = \emptyset$. Then $\{A_j\}_{j<\omega}$ is locally finite in X_γ , since each $B_{m,k}$ eventually misses the A_j 's. The $x_{\gamma,i}$'s are then the only possible limits of the A_j 's. But $N(x_{\gamma,i})$ is disjoint from A_j for $j > i$. Thus the A_j 's are locally finite in X . Since the A_j 's are also closed disjoint, in fact the collection is discrete. \square

Note that by Lemma 5, Theorem 12 does not offer a roadmap for constructing a locally compact normal space with the MOP which is not paracompact.

We note, for future reference, that:

Corollary 13. *A countable topological sum of spaces satisfying the hypotheses of Theorem 12 also has the MOP.*

Proof. The sum also satisfies these hypotheses. \square

Theorem 14. *Suppose X is locally compact, locally countable, $|X| \geq 2^{\aleph_0}$, and every closed subspace of size 2^{\aleph_0} has the MOP. Then X has the MOP.*

Corollary 15. *CH implies if X is locally compact, locally countable, and closed subspaces of size \aleph_1 have the MOP, then so does X .*

Corollary 16. *CH implies if X is locally compact, locally countable, and closed subspaces of size \aleph_1 are paracompact, then X has the MOP.*

Corollary 17. *CH implies if X is locally compact, locally countable, countable subsets have countable closures, and each closed $Y \subseteq X$ of size \aleph_1 satisfies the conditions for X in Theorem 12, then X has the MOP.*

The first and third corollaries are immediate. The second is because local compactness is closed-hereditary, and locally compact, paracompact spaces have the MOP.

Proof of Theorem 14. Let M be a countably closed elementary submodel of size 2^{\aleph_0} containing the space X and a moving off collection \mathcal{A} for it. By first countability, $X \cap M$ is a closed subspace of X , so it will suffice to find

a discrete collection $\{A_n\}_{n<\omega}$ included in \mathcal{A} , with each $A_n \subseteq X \cap M$, and $\{A_n\}_{n<\omega}$ discrete in $X \cap M$. It suffices to show $\mathcal{A} \cap M$ is moving off for $X \cap M$. Let F be a compact subspace of $X \cap M$. Since compact sets are countable and M is countably closed, $F \in M$. Then, since $M \models \mathcal{A}$ is moving off, $M \models (\exists A \in \mathcal{A})(F \cap A = \emptyset)$. But then there is an $A \in \mathcal{A} \cap M$ such that $F \cap A = \emptyset$. \square

Our previous counterexamples were not \aleph_1 -collectionwise Hausdorff; now we can get one that satisfies that property:

- f) A consistent-with-CH example of a locally compact, normal, \aleph_1 -collectionwise Hausdorff space with the MOP which is not paracompact.

A ladder system space X on a non-reflecting stationary set E of ω -cofinal ordinals in ω_2 is easily seen to be \aleph_1 -collectionwise Hausdorff, because initial segments of E are non-stationary. In fact, subspaces of size $\leq \aleph_1$ are paracompact, and hence such small closed ones have the MOP. X is not paracompact because it is not \aleph_2 -collectionwise Hausdorff. Shelah [20] forced to make X normal, consistent with CH. \square

- g) A Souslin tree with the usual tree topology is collectionwise normal [7]. It has countable extent but is not Lindelöf, so is not paracompact. By Theorem 12 it has the MOP.

A similar proof of the MOP works for any other ω_1 -tree with the tree topology, but normal ones that are not paracompact will not be found in ZFC - see [7]. Gruenhage [10] proved earlier that any Aronszajn tree has the MOP.

4 More results in a model of $\text{PFA}(S)[S]$

There are some easy observations about the MOP in the model of $\text{PFA}(S)[S]$ we have mentioned earlier.

Theorem 18. *In the model of Lemma 3, Theorem 9, etc., locally compact, hereditarily normal, countably tight spaces with the MOP are paracompact.*

Proof. In a countably tight space, countably compact subspaces are closed [3]. Closed subspaces of a space satisfying the MOP also satisfy it. Perfect pre-images of ω_1 are countably compact but not compact. Now apply Lemmas 3 and 6. \square

Corollary 19. *In this model, first countable hereditarily normal spaces satisfying the MOP are paracompact.*

Proof. By Theorem 18 and Lemma 7. □

Theorem 20. *In this model, locally compact, normal, countably tight spaces with Lindelöf number $\leq \aleph_1$ satisfying the MOP are paracompact.*

Proof. They do not include a perfect pre-image of ω_1 , so we can apply Lemma 5. □

Corollary 21. *In this model, first countable, normal spaces with Lindelöf number $\leq \aleph_1$ satisfying the MOP are paracompact.*

Proof. Apply Lemma 7 and Theorem 20. □

Corollary 22. *In this model, locally compact, normal, countably tight spaces satisfying the MOP (in particular, first countable normal spaces satisfying the MOP) are paracompact, provided countable sets have Lindelöf closures.*

Proof. In [24] it is shown that in this model,

Lemma 23. *In this model, locally compact normal spaces not including a perfect pre-image of ω_1 are paracompact, provided countable sets have Lindelöf closures.* □

5 Baire powers of function spaces

Definition. A space is weakly α -favorable [1] if Nonempty has a winning strategy in the Banach-Mazur game. In that game, players take turns picking an open set included in their opponent's pick. The first player, Empty, wins if, after ω plays, the intersection of the open sets is empty; otherwise the second player, Nonempty, wins.

Lemma 24 [16]. *A locally compact X is paracompact if and only if $C_k(X)$ is weakly α -favorable.*

Galvin and Scheepers [9] note that White [25] showed that all box powers of weakly α -favorable spaces are Baire, and then prove:

Theorem 25. *If it is consistent there is a proper class of measurable cardinals, then it is consistent that if all box powers of a space are Baire, then the space is weakly α -favorable.*

They then ask whether there are any consistent counterexamples. Let us consider the particular case of $C_k(X)$ for X locally compact. Their result then entails:

Corollary 26. *If it is consistent there is a proper class of measurable cardinals, then it is consistent that if all box powers of $C_k(X)$ are Baire, where X is locally compact, then X is paracompact.*

Scheepers pointed out to me that Oxtoby [19] proved that any product of Baire spaces with a countable base is Baire, but that a Bernstein set of reals is Baire but not weakly α -favorable, so in the Theorem, ordinary powers are not enough.

In fact, they are not even sufficient for the Corollary. Example b) is a counterexample:

Theorem 27. *Suppose X satisfies the hypotheses of Theorem 12. Then arbitrary powers of $C_k(X)$ are Baire.*

Proof. Fleissner and Kunen [8] prove

Lemma 28. *Let $\kappa \geq \omega$. If X^ω is Baire, then X^κ is Baire.*

McCoy and Ntantu [17] prove

Lemma 29. *Let $\bigoplus_{\alpha < \lambda} X_\alpha$ be the topological sum of copies of X . Then $C_k(\bigoplus_{\alpha < \lambda} X_\alpha)$ is homeomorphic to $(C_k(X))^\lambda$.*

Thus, by Corollary 13, our assertion that b) is a counterexample is verified. \square

The preceding two lemmas prove that:

Theorem 30. *If countable sums of copies of a locally compact X have the MOP, then arbitrary sums of copies of X have the MOP.*

Surprisingly, there is a consistent example of locally compact spaces X and Y , each having the MOP, but $X \oplus Y$ does not have the MOP [16].

We do have one necessity theorem for large cardinals, but do not know whether the hypothesis is vacuous:

Theorem 31. *Suppose that whenever all usual powers of $C_k(X)$ are Baire, for locally compact, \aleph_1 -collectionwise Hausdorff X , then X is paracompact. Then it is consistent that there is a strong cardinal.*

Proof. Take a non-reflecting stationary set E of ω -cofinal ordinals in λ^+ , for some $\lambda \geq \mathfrak{c}$. It is known (attributed to R. Jensen) that if it's consistent no such set exists, then it's consistent there is a strong cardinal - see [12]. Form a ladder system space X on E . X is not paracompact, but initial segments of it are. Consider an arbitrary sum $\bigoplus_{\alpha \in S} X_\alpha$ of copies of X . I claim that $\bigoplus_{\alpha \in S} X_\alpha$ has the MOP, whence $[C_k(X)]^{|S|}$ is Baire. By Corollary 15, it suffices to show closed subspaces of $\bigoplus_{\alpha \in S} X_\alpha$ of size $\leq 2^{\aleph_0}$ have the MOP. But they are all paracompact, so they do. But X is not paracompact. \square

A strong cardinal (see [13] for the definition) has arbitrarily large measurable cardinals below it. Thus, in V_κ , where κ is the least strong cardinal, there is a proper class of measurable cardinals, but no strong cardinal in an inner model. Collapsing these cardinals as in [9] yields a model in which there is a ladder system space X on a non-reflecting stationary set as above. Some box power of $C_k(X)$ is then not Baire.

Large cardinals can be used to destroy non-reflecting stationary sets; this translates into results about small subspaces being paracompact implying the whole space is paracompact. For example:

Theorem 32 [23]. *Martin's Maximum implies that if a first countable space is either generalized ordered or monotonically normal and closed subspaces of size \aleph_1 are paracompact, then the space is paracompact.*

For more results of this sort, see [23].

In [5], Fleissner raises the question of whether, if the box product of a collection of Baire spaces is Baire, its Tychonoff product is Baire. Also see [8]. The converse is not true [8]. Note that for box powers, in the model of Galvin and Scheepers, this is true, since Tychonoff products of weakly α -favorable spaces are Baire [25]. Fleissner also asks whether the box product of Baire spaces with a countable base is Baire [5]. In this model, this is not true - consider the box powers of a Bernstein set.

One might be tempted, in view of the countable nature of the Baire Category Theorem and of weak α -favorability, to conjecture that the Baireness of countable box powers would consistently be sufficient to imply weak α -favorability, at least for spaces with a countable base. This is not true. L. Zsilinszky [26] proved:

Theorem 33. *The countable box power of a Baire space with a countable base is Baire.*

But again, a Bernstein set is not weakly α -favorable.

6 Problems

The most interesting open question in this area is raised in [11]:

Problem 1. *Is there in ZFC a locally compact, normal space with the MOP which is not paracompact (equivalently, $C_k(X)$ is not weakly α -favorable)?*

None of the examples we have mentioned exist in the model of $\text{PFA}(S)[S]$ we have been using, since in that model there are no Souslin trees, and normal first countable spaces are collectionwise Hausdorff [14].

Problem 2 ([9]). *Are large cardinals necessary for Theorem 25?*

Problem 3. *Can one prove in ZFC that some box power of C_k of a ladder system space on a (non-reflecting?) stationary set of ω -cofinal ordinals is not Baire?*

Problem 4 ([5]). *Can one prove in ZFC that if a box product of a collection of Baire spaces is Baire, then its Tychonoff product is Baire?*

References

- [1] CHOQUET, G. *Lectures on Analysis, vol. I*. Benjamin, New York, 1969.
- [2] DEVLIN, K. J., AND SHELAH, S. A note on the normal Moore space conjecture. *Canad. J. Math.* 31 (1979), 241–251.
- [3] EISWORTH, T., AND NYIKOS, P. J. Antidiamond principles and topological applications. *Trans. Amer. Math. Soc.* 361 (2009), 5695–5719.
- [4] EKLOF, P. Set theory generated by Abelian group theory. *Bull. Symb. Logic* 3 (1996), 1–16.
- [5] FLEISSNER, W. G. Box products of Baire spaces. *General topology and its relations to modern analysis and algebra, IV (Proc. Fourth Prague Topological Sympos., Prague 1976), Part B*, 125–126.

- [6] FLEISSNER, W. G. When is Jones' space normal? *Proc. Amer. Math. Soc.* **46** (1974), 294–298.
- [7] FLEISSNER, W. G. Remarks on Souslin properties and tree topologies. *Proc. Amer. Math. Soc.* **80** (1980), 320–326.
- [8] FLEISSNER, W. G., AND KUNEN, K. Barely Baire spaces. *Fund. Math.* **101** (1978), 229–240.
- [9] GALVIN, F., AND SCHEEPERS, M. Baire spaces and infinite games. ArXiv:1401.6061 v2, 24 Oct. 2014.
- [10] GRUENHAGE, G. The story of a topological game. *Rocky Mountain J. Math* **36** (2006), 1885–1914.
- [11] GRUENHAGE, G., AND MA, D. Baireness of $C_k(X)$ for locally compact X . *Topology Appl.* **80** (1997), 131–139.
- [12] JECH, T. Stationary sets. In *Handbook of Set Theory*, A. Kanamori and M. Foreman, Eds. Springer, Heidelberg, 2010, pp. 93–128.
- [13] KOELLNER, P., AND WOODIN, W. Large cardinals from determinacy. In *Handbook of Set Theory*, A. Kanamori and M. Foreman, Eds. Springer, Heidelberg, 2010, pp. 1951–2120.
- [14] LARSON, P., AND TALL, F. D. Locally compact perfectly normal spaces may all be paracompact. *Fund. Math.* **210** (2010), 285–300.
- [15] LARSON, P., AND TALL, F. D. On the hereditary paracompactness of locally compact hereditarily normal spaces. *Canad. Math. Bull.* **57** (2014), 579–584.
- [16] MA, D. The Cantor tree, the γ -property, and Baire function spaces. *Proc. Amer. Math. Soc.* **119** (1993), 903–913.
- [17] MCCOY, R. A., AND NTANTU, I. *Topological properties of spaces of continuous functions*. No. 1315 in Lect. Notes in Math. Springer, New York, 1988.
- [18] OSTASZEWSKI, A. On countably compact, perfectly normal spaces. *J. London Math. Soc.* **14**, 2 (1976), 505–516.

- [19] OXToby, J. C. Cartesian products of Baire spaces. *Fund. Math.* 49 (1960/1961), 157–166.
- [20] SHELAH, S. Remarks on λ -collectionwise Hausdorff spaces. *Topology Proc.* 2 (1977), 583–592.
- [21] SHELAH, S. Whitehead groups may not be free, even assuming CH, I. *Israel J. Math.* 28 (1977), 193–204.
- [22] TALL, F. D. Normality vs collectionwise normality. In *Handbook of Set-Theoretic Topology*, K. Kunen and J. Vaughan, Eds. North-Holland, Amsterdam, 1984, pp. 685–732.
- [23] TALL, F. D. Topological applications of generic huge embeddings. *Trans. Amer. Math. Soc.* 341 (1994), 45–68.
- [24] TALL, F. D. $\text{PFA}(S)[S]$ and locally compact normal spaces. *Topology Appl.* 162 (2014), 100–115.
- [25] WHITE, JR., H. E. Topological spaces that are α -favorable for a player with perfect information. *Proc. Amer. Math. Soc.* 50 (1975), 477–482.
- [26] ZSILINSZKY, L. Products of Baire spaces revisited. *Fund. Math.* 183 (2004), 115–121.